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DEMAND DURING LEAD TIME IN AN INVENTORY CONTROL SYSTEM

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DEMAND DURING LEAD TIME
IN AN INVENTORY CONTROL SYSTEM

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DEMAND DURING LEAD TIME
IN AN INVENTORY CONTROL SYSTEM

[redacted]

by

Robert L. Williams

Lieutenant, Supply Corps, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

IN

OPERATIONS RESEARCH

United States Naval Postgraduate School

Monterey, California

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ABSTRACT

Demand during lead time is considered as the sum of a random number of random variables. Discrete random variables are considered as describing demand and lead time and probability generating functions are used to determine the probability generating function of demand during lead time. The probability distribution of demand during lead time is extracted from the probability generating function by using a power series expansion and the calculation of probabilities from the expansion are discussed. The mean and variance of demand during lead time are also obtained from the probability generating function.

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TABLE OF SYMBOLS

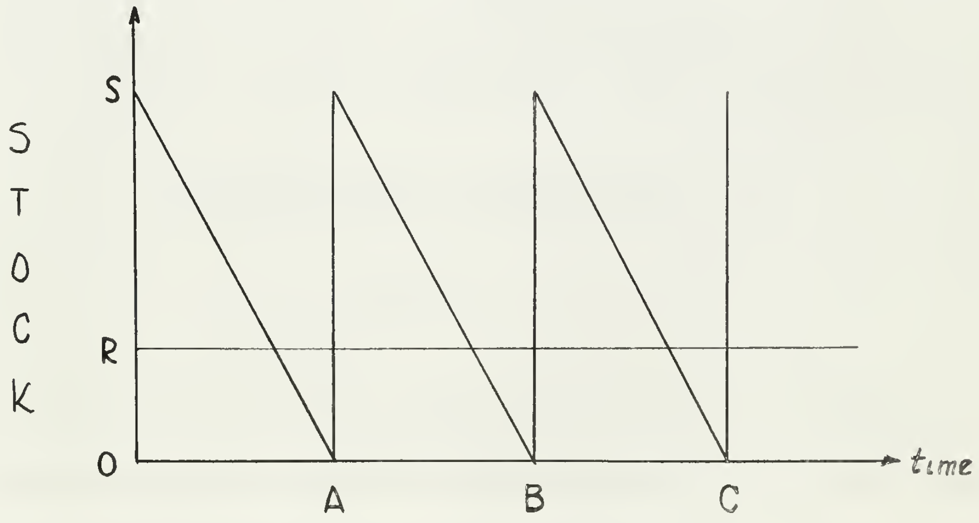
1.	D	Random variable denoting demand, upper case letter denotes random variable
2.	$\bar{D}$	Expected value or mean of demand
3.	L	Random variable denoting lead time
4.	$\bar{L}$	Expected value or mean of lead time
5.	R	Reorder point
6.	VSL	Variable safety level
7.	Z	Random variable denoting demand during lead time
8.	$\bar{Z}$	Expected value or mean of demand during lead time
9.	$f(s)$	Probability generating function of demand
10.	$g(s)$	Probability generating function of lead time
11.	$h(s)$	Probability generating function of demand during lead time
12.	$E(X)$	Expected value of the random variable X
13.	σ_X^2	Denotes variance of the random variable X
14.	λ	Parameter of the Poisson probability law
15.	r,p	Parameters of the negative binomial probability law
16.	$p(x;r,p)$	Denotes the probability mass function of X with parameters r and p
17.	$f(t;u,k)$	Denotes the probability density function of T with parameters u and k
18.	x	Lower case letters denote a specific value of a random variable X

1. Introduction

The two major problems in any inventory control system are to decide upon the amount to order each time replenishment is necessary and to decide what should be the minimum stock level, of any item, at which an order should be placed. The second problem is the subject of this paper.

The minimum stock level at which an order for replenishment of an item is the reorder point, denoted by R. The reorder point is a function of the demand and the lead time of an item. The demand for an item is the rate at which the stock of the item is depleted. The lead time of an item is the length of the interval from the time a replenishment order is placed until the order is in stock, ready to satisfy the customer's demand.

A simple deterministic model of an inventory system with constant demand and lead time is shown in figure 1.

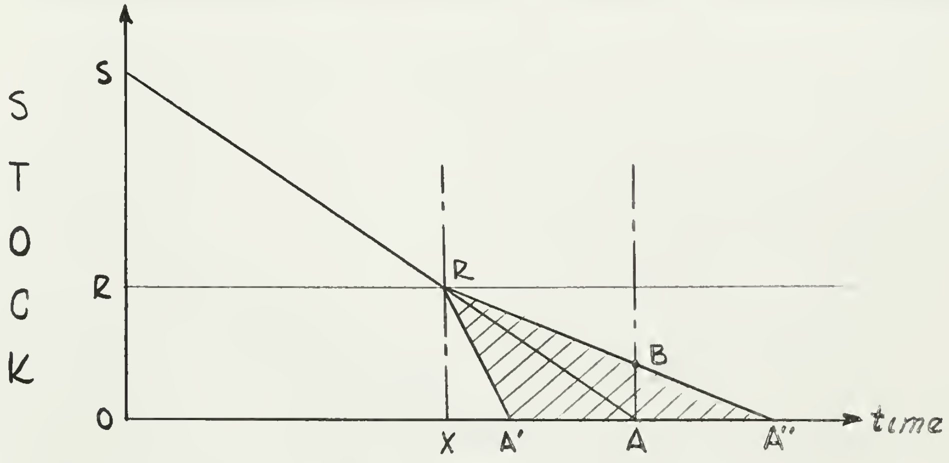


Deterministic Inventory Model

Figure 1

The stock is depleted at a uniform rate from a stock level of S until it reaches the reorder point. When the stock level of R units is reached, a replenishment order for S units is released and the order is received at time A, just as the stock is exhausted. The stock replenishment raises the stock level back to S units and the process is repeated.

Rarely is demand uniform, however, and there is uncertainty about when the R units of stock on hand at the time of placing an order will be exhausted. A more realistic model of an inventory system with variable demand and constant lead time is shown in figure 2.



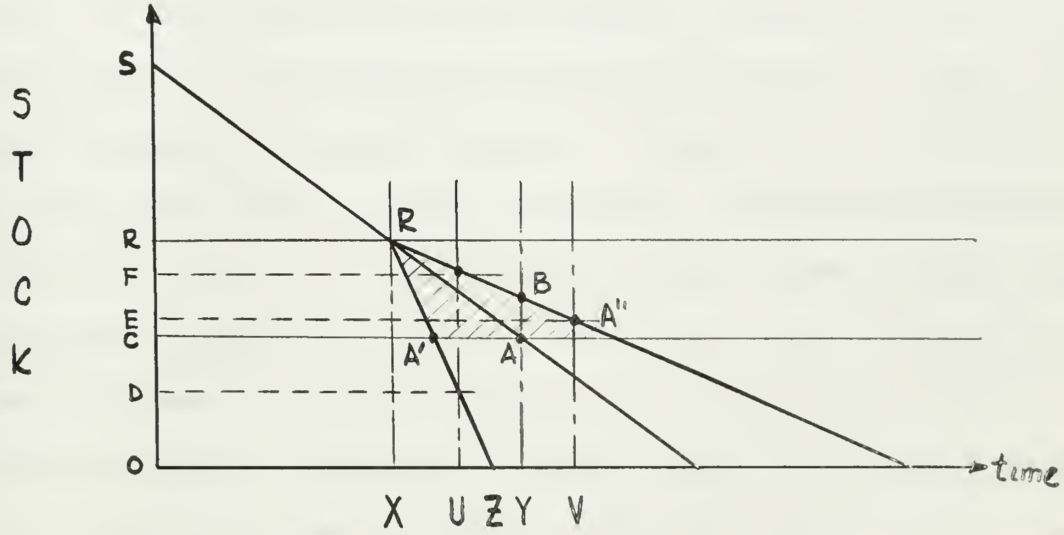
Variable Demand Inventory Model

Figure 2

S, R and A are the same in figure 2 as they were in figure 1. The expected depletion of stock is along the line S A. If the lead time was deterministic, an order placed at R, time X, would be received at time A. In the time interval during which replacement stock is an order, X A, there can be a variation in the demand;

therefore, the demand may deplete the R units of stock along any line between a maximum demand, R A', and a minimum demand, R A", shown by the shaded area A' R A". The variation in demand allows for the possibility of depleting the stock at time A' and not being able to meet demands for stock in the period A' A. There is also the possibility that the demand could be as low as R A" and in this case there would be B units of stock on hand when the replacement order is received.

A third and the most realistic model, considering both variable demand and lead time, is shown in figure 3. The first two models were special cases of this model and therefore model 3 will serve to describe the inventory system considered throughout the rest of this paper.



Variable Demand and Lead Time Model

Figure 3

The variation in the lead time will cause the replenishment time interval X Y to vary. Time X is the time at which the reorder

point, R, is reached, but it is no longer certain that the order will be received at time Y. The order may be received at time U or time V or any time in between, if we let time U and time V represent the minimum and maximum times of receipt of the order. Figure 3 is similar to figure 2, but the zero stock level of figure 2 is the same number of units below R as C is in figure 3. Figure 3 then is the same diagram except that the entire stock is increased by C units to allow for the variation in lead time as well as the variation in demand. Let us consider the four possible extreme cases of maximum and minimum demand and lead time. A replenishment order is placed when the stock on hand is R units at time X. If the maximum demand and the minimum lead time occur, the stock will be depleted along line R A' until there are D units at time U when the replacement stock is received. If the minimum demand and the maximum lead time occur, the stock will be depleted along line R A" until there are E units in stock at time V when the order is received. Minimum demand and minimum lead time will result in F units of stock on hand at time U when the order is received. In each of these three cases, R units of stock have been more than enough to provide protection against stockouts with both variable demand and lead time. Only in the case of maximum demand and minimum lead time would the stock be below C units. The last case is the combination of maximum demand and maximum lead time. The stock will be depleted along the maximum demand line R A' and will be exhausted at time Z. The order will not be received until the maximum lead time, time V, therefore the system will be out of stock for a period of V-Z time units.

From figure 3, it can be seen that if the demand follows the line R A and time Y is the lead time, then the demand during lead time is equal to R - C units of stock. It can also be seen in figure 3 that as the height O C is increased the intersection of the actual demand and the actual lead time, whatever they may be, will intersect above the zero stock level more often than below. The reorder point, R, is then the sum of the expected demand during lead time plus a safety level or buffer stock of C units. The reorder point equation is then;

$$R = \bar{\bar{Z}} + V\,S\,L \tag{1.1}$$

where

R = reorder point

$\bar{\bar{Z}}$ = expected demand during lead time

$V\,S\,L$ = variable safety level

The safety level is denoted as a variable safety level because the protection against a stockout increases as the number C is increased. The safety level can be varied so that no stockouts occur or some small number of stockouts occur in a specified number of replenishment periods. That is, the inventory manager may decide that a safety stock large enough to protect against no stockouts would be too expensive and therefore he is willing only to maintain a safety stock that will protect against stockouts ninety or ninety-five percent of the time. The safety level is also a function of the reorder quantity since the larger the reorder quantity for a given lead time the less frequently the system is exposed to a stockout. This

consideration is taken into account when the protection level is determined and does not affect the reorder point equation [6], [1].

2. Discussion of the Problem

The availability of alternative courses of action implies the need for managers to make decisions. The selection of the reorder point for an item in the manager's stock is then a decision problem. The reorder point rule must satisfy an objective function (such as maximum service) subject to some constraints imposed by higher authority, available funds, available facilities, etc. In almost all cases, the method by which the problem is solved requires a model of the operations. Such a model is based on a set of relations linking the important variables. Figure 3 is the geometric model of the inventory system being considered and the reorder point equation (1.1) will lead to the mathematical model which will be used to determine the probability distribution function of demand during lead time.

The reorder point equation is a rule by which the inventory manager may decide when to reorder replacement stock. The variable $\bar{Z}$ is the mean of the probability distribution of the demand during lead time and the variable V S L is a function of the variance of the same probability distribution. Therefore; in order to use the reorder point equation, the probability distribution function of demand during lead time or at least the mean and variance of the distribution function must be known.

Let Z be a random variable which describes the demand during lead time, then the number of demands that occur during a lead time may be written as:

$$Z = D_1 + D_2 + \dots + D_L \qquad (2.1)$$

where

Z = the number of items demanded during a lead time

D_i = demands , $i = 1, \ldots, L$

L = lead time

It is assumed that the demands are independent and identically distributed and demand and lead time are independent discrete random variables with known probability mass functions. In general, this paper will only consider discrete random variables since demands naturally occur at discrete intervals and lead times are normally described in days, weeks or months. Equation (2.1) is then a model which describes the demand during lead time by linking together the random variables D and L and is a model that can be used to determine the expected demand during lead time, $\overline{\overline{Z}}$, and the variable safety level, V S L. Equation (2.1) can be described more concisely as:

$$Z = \sum_{i=1}^{L} D_i \qquad (2.2)$$

Using equation (2.2) as the model of demand during lead time, it will be shown that the mean and variance can be computed for any probability mass functions that describe demand and lead time or from the mean and variance of demand and lead time if that is all that is known about them. It will also be shown that the distribution of demand during lead time can be found, in the discrete case, by the use of probability generating functions. After the probability distribution is found, it will then be shown that it

can be used to set the reorder point using a variable safety level; given the level of customer service desired. The final section treats a special case in which lead time is a continuous random variable and the resulting distribution of demand during lead time has a familiar negative binomial form.

3. The Mean and Variance of Demand During Lead Time

The mean and variance of Z are derived using general probability mass functions for D and L.

Let:

p (d) be the mass function of demand with mean $\bar{D}$ and variance σ_D^2

p (l) be the mass function of lead time with mean $\bar{L}$ and variance σ_L^2

p (z) be the mass function of demand during lead time with mean $\bar{Z}$ and variance σ_Z^2

The mean of Z is the expected value of Z and is denoted as $\bar{Z}$

$$\begin{aligned} E(Z) &= E\left[E(Z|L)\right] \\ &= E\left[L\ E(D)\right] \\ &= E(D)\ E(L) \\ &= \bar{D}\ \bar{L} \end{aligned}$$

Therefore the expected demand during lead time is

$$\bar{Z} = \bar{D}\ \bar{L}$$

The variance of demand during lead time is derived as follows:

$$Var(Z) = E\left[Var(Z|L)\right] + Var\left[E(Z|L)\right]$$

$$Var(Z) = E\left[L\ \sigma_D^2\right] + Var\left[L\ E(D)\right]$$

$$= E(L)\,\sigma_D^2 + \left[E(D)\right]^2 \sigma_L^2$$

$$= \bar{L}\,\sigma_D^2 + \bar{D}^2\,\sigma_L^2$$

The mean and variance of demand during lead time are functions of the means and variances of the demand and lead time only, therefore a knowledge of the mass functions of demand and lead time is sufficient to calculate the mean and variance of demand during lead time. They may also be calculated empirically using historical demand and lead time data to estimate the means and variances of demand and lead time. The mean and variance of a distribution function may also be calculated from the probability generating function of the distribution. The probability generating function of a probability distribution function is discussed in the next section and the calculation of the mean and variance from a probability generating function is shown in Appendix I.

4. Distribution of Demand During Lead Time

In this section, the probability generating function of demand during lead time will be derived in general. The generating function will then be used to obtain the probability distribution of lead time demand by expanding the generating function in a power series and extracting the coefficients. Probability generating functions are defined for discrete random variables only. Therefore; the discussion will be limited to discrete random variables.

The probability generating function of demand during lead time is the generating function of the sum of a random number of random variables and is defined as

$$h(s) = E(s^{Z}) = \sum_{k=0}^{\infty} s^{k}\, P\left[Z=k\right]$$

To derive h(s), let the probability generating function of demand and lead time be

$$g(s) = E(s^{L})$$

$$f(s) = E(s^{D})$$

and

$$h(s) = E(s^{Z}) = E\,(s^{Z} \mid L = l)$$

now

$$\begin{aligned} E(s^{Z} \mid L = l) &= E(s^{D_1}\ s^{D_2} \ldots s^{D_l}) \\ &= E(s^{D_1})E(s^{D_2})\ldots E(s^{D_l}) \\ &= \left[E(s^{D})\right]^{l} \end{aligned}$$

since the D_i were assumed independent and identically distributed. Also since $f(s) = E(s^D)$, we have

$$E(s^Z | L = 1) = \left[f(s)\right]^1$$

and

$$E(s^Z | L) = \left[f(s)\right]^L$$

$$E\left[E(s^Z | L)\right] = E\left\{\left[f(s)\right]^L\right\}$$

$$= g\left[f(s)\right]$$

since

$$g(s) = E(s^L)$$

$$\therefore h(s) = g\left[f(s)\right]$$

Thus, if the distribution functions of demand and lead time are known or assumed, the probability generating functions of demand during lead time can be derived. Then with a knowledge of h(s), the probability distribution of demand during lead time is known. Examples showing the computation of the distribution of Z will be given later.

Implementation of the model given in equation (2.2) may generally take place using familiar discrete distributions for the variables, since such distributions have convenient mathematical properties. These familiar distributions usually have positive probability at zero and this situation usually is not compatible with lead time. Practically, lead time can never be zero, if for no other reason than it takes some time to prepare an order. On the other hand, if the data

in the inventory control system is based on the month as the basic time unit, a lead time of one day could be considered zero. In any event, if the mean of the lead time distribution is small, then the use of a popular distribution will result in probability at zero which is not negligible. Mathematically, the model is correct since when L equals zero the sum is empty and Z is zero. Demand during lead time can also be zero when demand is zero during the lead time or when both demand and lead time are zero.

There are several methods by which the input distribution can be adapted to consider either allowing or not allowing some probability for lead time to be zero. In the first case, when lead time can be zero, the straight forward use of the model with distribution functions of demand and lead time that are defined on the non-negative integers will have provided the desired results. There are two methods that may be used to prevent the consideration of zero lead time in the model. These methods redistribute the probability mass functions from a range over the non-negative integers to a range over the positive integers. The first is translation, i.e., let $L = 1+X$, where X takes non-negative integral values; and the second is truncation, i e., use a frequency function that is conditioned on the variable being positive. Let us consider the Poisson distribution as an example. The Poisson distribution is usually defined as:

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad , \text{ for } \lambda > 0, \ x = 0,1,2,\ldots$$

The mass function can be translated over the positive integers by

writing it as:

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \quad , \text{ for } \lambda > 0,\ x = 1,2, \ldots$$

or the distribution function can be truncated by using a conditional probability. That is:

$$P\left[X = x \middle| X > 0\right] = \frac{e^{-\lambda}\ \lambda^{x}}{x!(1-e^{-\lambda})}$$

$$= \frac{1}{e^{\lambda}-1}\ \frac{\lambda^{x}}{x!}$$

$$\therefore p(x;\lambda) = \frac{\lambda^{x}}{(e^{\lambda}-1)\ x!} \quad , \text{ for } \lambda > 0,\ x = 1,\ 2, \ldots$$

There are then three choices offered for adjusting the input to fit the situation or the preference of the user.

Let us now consider some generating functions of demand and lead time so that they will be available for use in determining the probability generating function of Z. The Poisson and negative binomial distributions have frequently served as good approximations for the distribution of demand so they will be used in the following examples. Lead time distributions have not been as well documented, but the negative binomial distribution is a good model for discrete waiting time random variables and when the parameter r equals one, the negative binomial becomes the geometric distribution. As a second lead

time distribution, the Poisson distribution will be used. The use of the Poisson distribution for lead time may not be a good example, but with a large mean, the Poisson distribution roughly approximates a normal distribution with the mean equal to the variance [1].

The generating functions for the Poisson and negative binomial distributions in each of the three cases are given below.

For the Poisson, when

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad , x = 0, 1, \ldots$$

The probability generating function is:

$$f(s) = E(s^{X}) = \sum_{k=0}^{\infty} s^{k}\, P[X = k]$$

$$= e^{\lambda(s-1)} \qquad (4.1)$$

when

$$p(x;\lambda) = \frac{e^{-\lambda}\,\lambda^{x-1}}{(x-1)!} \qquad , x = 1, 2, \ldots$$

$$f(s) = s\, e^{\lambda(s-1)} \qquad (4.2)$$

when

$$p(x;\lambda) = \frac{\lambda^{x}}{(e^{\lambda}-1)\, x!} \qquad , x = 1, 2, \ldots$$

$$f(s) = \frac{e^{\lambda s} - 1}{e^{\lambda} - 1}$$

for negative binomial distribution, when

$$p(x;r,p) = \binom{r+x-1}{x} p^r q^x \qquad , x = 0, 1, \ldots$$

$$f(s) = \left(\frac{p}{1-q\ s}\right)^r \qquad (4.3)$$

when the distribution is translated

$$p(x;r,p) = \binom{r+x-2}{x-1} p^r q^{x-1} \qquad , x = 1, 2, \ldots$$

$$f(s) = s\left(\frac{p}{1-qs}\right)^r$$

when the distribution is truncated

$$p(x;r,p) = \frac{\binom{r+x-1}{x} p^r q^x}{1-p^r} \qquad , x = 1, 2, \ldots$$

$$f(s) = \frac{p^r\left[(1-qs)^{-r}-1\right]}{1-p^r}$$

As an example of determining the probability generating function of demand during lead time, let

D be a Poisson random variable

L be a translated negative binomial

then

$$f(s) = e^{\lambda(s-1)}$$

and

$$h(s) = g\left[f\ \ (s)\right]$$

$$= p^r \sum_{k=1}^{\infty} e^{\lambda(s-1)k} \binom{r+k-2}{k-1} q^{k-1}$$

$$= p^r e^{\lambda(s-1)} \sum_{k=1}^{\infty} e^{\lambda(s-1)(k-1)} \binom{r+k-2}{k-1} q^{k-1}$$

$$= p^r e^{\lambda(s-1)} \sum_{k=1}^{\infty} \binom{r+k-2}{k-1} \left[q e^{\lambda(s-1)}\right]^{k-1}$$

$$= p^r e^{\lambda(s-1)} \left[1-q e^{\lambda(s-1)}\right]^{-r}$$

The probability generating functions of demand during lead time, Z, were derived since they can be used to find the probability of demand during lead time. That is, since

$$h(s) = E(s^Z) = \sum_{k=0}^{\infty} s^k P\left[Z = k\right]$$

Then in the power series expansion of h(s), the coefficient of s^k is the $P\left[Z = k\right]$. For example, writing out several terms of h(s), we have:

$$h(s) = P\left[Z = 0\right] + s\, P\left[Z = 1\right] + s^2\, P\left[Z = 2\right] + \ldots$$

It therefore is necessary to put h(s) in the form of a power series in order to determine the probability of demand during lead time for any k.

Using the probability generating functions for the variable Z

in the previous examples, we can put them in a more useful form. For Poisson demand and translated negative binomial lead time

$$h(s) = e^{\lambda(s-1)} \left[\frac{p}{1-qe^{\lambda(s-1)}} \right]^{r}$$

$$= e^{\lambda(s-1)} p^{r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} \left[qe^{\lambda(s-1)} \right]^{k}$$

$$= p^{r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} q^{k} e^{s\lambda(k+1) - (k+1)}$$

$$= p^{r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} q^{k} e^{-\lambda(k+1)} \sum_{n=0}^{\infty} s^{n} \frac{[\lambda(k+1)]^{n}}{n!}$$

$$= \sum_{n=0}^{\infty} s^{n} p^{r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} q^{k} e^{-\lambda(k+1)} \frac{[\lambda(k+1)]^{n}}{n!}$$

$\therefore P[Z = n]$ is the coefficient of s^n in the power series and

$$P[Z = n] = p^{r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} q^{k} e^{-\lambda(k+1)} \frac{[\lambda(k+1)]^{n}}{n!}$$

$$= p^{r} e^{-\lambda} \frac{\lambda^{n}}{n!} \sum_{k=0}^{\infty} \binom{r+k-1}{k} (qe^{\lambda})^{k} (k+1)^{n}$$

The calculation of these probabilities can be reduced if the sum from zero to infinity is reduced to a finite number of terms with a very small remainder. A method for finding the number of terms to

be summed until the remainder is negligible will be shown along with a computing technique for determining the probabilities.

To find a bound for the remainder of the series, let

$$A_K = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{K} \binom{r+k-1}{k} (qe^{-\lambda})^k (k+1)^n$$

$$a = qe^{-\lambda}$$

and

$$R_K = \sum_{k=K+1}^{\infty} u_k = u_{K+1} \sum_{l=0}^{\infty} \frac{u_{K+1+l}}{u_{K+1}}$$

then

$$\frac{u_{K+1+l}}{u_{k+1}} = \frac{\binom{r+k+l}{k+1+l} a^{k+1+l} (k+2+l)^n}{\binom{r+k}{K+1} a^{K+1} (k+2)^n}$$

$$= \frac{(r+K+l)!(K+1)!}{(K+1+l)!(r+K)!} \left(\frac{K+2+l}{K+2}\right)^n a^l$$

$$= \frac{(r+K+l)(r+K+l-1)\ldots(K+l+r-(r-2))}{(K+r)(K+r-1)\ldots(K+r-(r-2))} \left(\frac{K+2+l}{K+2}\right)^n a^l$$

$$= \frac{(r+k+l)(r+K+l-1)\ldots(K+l+2)}{(K+r)(K+r-1)\ldots(K+2)} \left(\frac{K+l+2}{K+2}\right)^n a^l$$

Dividing term by term

$$= \left(1 + \frac{1}{K+r}\right)\left(1 + \frac{1}{K+r-1}\right) \cdots \left(1 + \frac{1}{K+2}\right)\left(1 + \frac{1}{K+2}\right)^{n} a^{l}$$

$$\leqslant \left(1 + \frac{1}{K}\right)^{n+1} a^{l}$$

then

$$R_K \leqslant u_{K+1} \sum_{l=0}^{\infty} \left(1 + \frac{1}{K}\right)^{n+1} a^{l}$$

It is possible to dominate the right hand side with a integral provided the integrand is monotone decreasing. This will be true if K is sufficiently large. Thus, let

$$-B \equiv \ln a = -\lambda + \ln q$$

and

$$\gamma(l) = \left(1 + \frac{1}{K}\right)^{n+1} e^{-Bl}$$

then

$$\gamma'(l) = \left(1 + \frac{1}{K}\right)^{n} e^{-Bl} \left[-\left(1 + \frac{1}{K}\right) B + \frac{n+1}{K}\right]$$

Setting the derivative equal to zero, we have

$$l = \frac{n+1}{B} - K$$

∴ the maximum for l occurs if $K > \frac{n+1}{B}$

hence we must use K such that it is at least this large.

Now for $K > \frac{n+1}{B}$

$$\sum_{l=0}^{\infty} \left(1 + \frac{l}{K}\right)^{n+1} a^{l} = 1 + \sum_{l=1}^{\infty} \left(1 + \frac{l}{K}\right)^{n+1} a^{l}$$

$$\leqslant 1 + \int_{o}^{\infty} \left(1 + \frac{l}{K}\right)^{n+1} e^{-Bx} \, dx$$

$$= 1 + \sum_{j=0}^{n+1} \binom{n+1}{j} \left(\frac{1}{K}\right)^{j} \int_{o}^{\infty} x^{j} e^{-Bx} \, dx$$

making a change of variables, let v=Bx, then dv=Bdx, and

$$\sum_{l=0}^{\infty} \left(1 + \frac{l}{K}\right)^{n+1} a^{l} \leqslant 1 + \sum_{j=0}^{n+1} \binom{n+1}{j} \left(\frac{1}{K}\right)^{j} \frac{1}{B^{j+1}} \int_{o}^{\infty} v^{j} e^{-v} \, dv$$

$$= 1 + \sum_{j=0}^{n+1} \binom{n+1}{j} \left(\frac{1}{K}\right)^{j} \frac{1}{B^{j+1}} \, j!$$

$$\leqslant 1 + \frac{(n+1)!}{B} \left(1 + \frac{1}{KB}\right)^{n+1}$$

which is a bounded function of K, thus

$$R_K \leqslant u_{K+1} \left[1 + \frac{(n+1)!}{B} \left(1 + \frac{1}{KB}\right)^{n+1}\right]$$

$$= \binom{r+K}{K+1} e^{-B(K+1)} (K+2)^{n} \left[1 + \frac{(n+1)!}{B} \left(1 + \frac{1}{BK}\right)^{n+1}\right]$$

now if $KB > n+1$

$$R_K \leqslant \binom{r+K}{K+1} \; e^{-B(K+1)} \; (K+2)^n \left[1 + \frac{(n+1)!}{B} \; e \right]$$

In practice, all of the constants and parameters in the above equation are known except K. K can be determined by setting the equation equal to a very small number and solving for K. The error in $P\ (Z=n)$ will then be less than that small number times $p^r \, e^{-\lambda} \, \frac{\lambda^n}{n!}$.

Now that the value of K has been approximated, the probability distribution of demand during lead time can be found as

$$P\left[Z = n\right] = e^{-\lambda} \; p^r \, \frac{\lambda^n}{n!} \sum_{k=0}^{K} \binom{r+k-1}{k} \, a^k \, (k+1)^n$$

In order to simplify the calculation of this number manually or in a computer, the technique of "nesting" will be shown.

$$\sum_{k=0}^{K} \binom{r+k-1}{k} \; a^k (k+1)^n = 1 + \binom{r}{1} \; a 2^n + \binom{r+1}{2} \; a^2 \; 3^n + \dots + \binom{r+K-1}{K} a^K (K+1)^n$$

$$= 1 + ra \; 2^n + \frac{r(r+1)}{2!} \; a^2 \; 3^n + \dots + \frac{r(r+1)\dots(r+K-1)}{K!} \; a^K (K+1)^n$$

$$= \left(1 + ra \; 2^n \left(1 + \frac{(r+1)}{2} \; a \; \frac{3}{2}^{\,n} \left(1 + \dots \left(1 + \frac{(r+K-1)}{K} a \left(\frac{K+1}{K}\right)^n \right)\right) \dots \right)\right)$$

This can then be written in the compact form.

Let

$$A_0 = \frac{r+K-1}{K} \; a \left(\frac{K+1}{K}\right)^n$$

and

$$B_0 = 1 + A_0$$

$$B_1 = 1 + A_1 B_0$$

$$\vdots$$

$$B_j = 1 + A_j B_{j-1}$$

where

$$A_j = \frac{r+K-j-1}{K-j} \; a \left(\frac{K-j+1}{K-j}\right)^n \text{ and } B_{-1} = 1$$

then

$$P\left[Z = n\right] = \left[1 + A_{K-1} B_{K-2}\right] e^{-\lambda} p^r \frac{\lambda^n}{n!}$$

As another example

Example 2

Let

D be a negative binomial random variable

L be a translated Poisson random variable

Then

$$f(s) = \left(\frac{p}{1-qs}\right)^r \qquad \text{from eq. (4.3)}$$

and

$$g(s) = s\, e^{\lambda(s-1)} \qquad \text{from eq. (4.2)}$$

Then

$$h(s) = f(s)\, g\left[f(s)\right] = f(s)\, e^{\lambda\left[f(s)-1\right]}$$

Let us again find the Maclaurin expansion

$$h(s) = \left(\frac{p}{1-qs}\right)^r \exp\left\{\lambda\left[\left(\frac{p}{1-qs}\right)^r -1\right]\right\}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} p^{r\,(k+1)} \, (1-qs)^{-r(k+1)}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} p^{r(k+1)} \sum_{n=0}^{\infty} \binom{r(k+1)+n-1}{n} (q\ s)^n$$

$$= \sum_{n=0}^{\infty} s^n \, e^{-\lambda} \, q^n \sum_{k=0}^{\infty} \binom{r(k+1)+n-1}{n} \frac{\lambda^k}{k!} p^{r\,(k+1)}$$

and

$$P\left[Z = n\right] = e^{-\lambda} \; q^n \, p^r \sum_{k=0}^{\infty} \binom{r(k+1)+n-1}{n} \frac{(\lambda\, p^r)^k}{k!}$$

To find a bound for the remainder of the series

Let

$$A_K = \sum_{k=0}^{K} u_k = \sum_{k=0}^{K} \binom{r(K+1)+n-1}{n} \frac{(\lambda\, p^r)^k}{k!}$$

and

$$R_K = \sum_{k=K+1}^{\infty} u_k \leqslant u_K \sum_{l=0}^{\infty} \frac{u_{K+1}}{u_K}$$

Then

$$u_k = \frac{\left[r(k+1)+n-1\right]!}{\left[r(k+1)-1\right]!\; n!} \; a^k \qquad , \text{ where } a = \lambda p^r$$

and

$$\frac{u_{K+1}}{u_K} = \frac{(r\,K+rl+n-1)!\ \ (rK+r-1)!}{(r\,K+rl-1)!\ (r\,K+r+n-1)!}\,a^l$$

$$= \frac{(rK+rl+r+n-1)\ (rK+rl+n-2)\ldots(rK+r+n)}{(rK+rl+r-1)\ (rK+rl+r-2)\ \ldots\ (rK+r)}\,a^l$$

$$\leqslant \left(1+\frac{n}{r(K+l+1)-1}\right)\left(1+\frac{n}{r(K+l+1)-2}\right)\ldots\left(1+\frac{n}{r(K+1)}\right)a^l$$

$$\leqslant \left(1+\frac{n}{r(K+1)}\right)^{rl} a^l$$

Thus

$$\frac{u_{K+1}}{u_K} \leqslant \left[\left(1+\frac{n}{r(K+1)}\right)^{r} a\right]^l$$

let

$$\gamma = \left[\left(1+\frac{n}{r(K+1)}\right)^{r} a\right]$$

We then have to choose minimum K such that $\gamma < 1$ so the resulting geometric series can be summed, i.e.,

$$\gamma = \left[\left(1+\frac{n}{r(K+1)}\right)^{r} a\right] < 1$$

if

$$1 + \frac{n}{r(K+1)} < a^{-\left(\frac{1}{r}\right)}$$

and

$$K > \frac{n \quad a^{\left(\frac{1}{r}\right)}}{r\left(1-a^{\left(\frac{1}{r}\right)}\right)} - 1$$

Then $\gamma < 1$

Now

$$R_K = u_K \sum_{l=0}^{\infty} \frac{u_{K+l}}{u_K} < u_K \sum_{l=0}^{\infty} \gamma^l$$

$$= u_K \frac{1}{1-\gamma}$$

and to solve for K we have

$$e^{-\lambda} q^n p^r u_K \left(\frac{1}{1-\gamma}\right) = \frac{[r\ (K+1)+n-1]\,!\quad p^{rK}\ e^{-\lambda} q^n\ p^r}{[r\ (K+1)\ -1]\,!\,n!\ \left[1-p^r\ \left(1+\frac{n}{rK}\right)^r\right]}$$

Upon setting this equation equal to an allowable error, K can be determined.

The estimate of the remainder is the value of the allowable error from which K was determined.

Then, given K

$$A_K = \sum_{k=0}^{K} \binom{r(k+1)+n-1}{n} \frac{(\lambda p^r)^k}{k!}$$

The "nesting" technique is helpful in computing the probabilities of demand during lead time.

$$P\left[Z = n\right] = e^{-\lambda} p^r q^n A_K \qquad , r \geqslant 1$$

and

$$A_K = \frac{(r+n-1)!}{(r-1)!n!} + \frac{(2r+n-1)!(\lambda p^r)}{(2r-1)!n!} + \frac{(3r+n-1)!(\lambda p^r)^2}{(3r-1)!\ n!} + \cdots$$

$$= \frac{(r+n-1)!}{(r-1)!n!}\left(1 + \frac{(2r+n-1)\ldots(r+n)\ (\lambda p^r)}{(2r-1)\ \ldots\ (r)}\right.\left(1 + \cdots\right.$$

$$\cdots\left(1 + \frac{(rk+n-1)\ \ldots\ (rk+n-r)\ (\lambda p^r)}{(rk-1)\ \ldots\ (rk-r)k}\right.\left(1 + \cdots\right.$$

$$\cdots\left(1 + \frac{(rK+n-1)\ \ldots\ (rK+n-r)\ (\lambda p^r)}{(rK-1)\ \ldots\ (rK-r)}\right)\Bigg)\cdots\Bigg)\Bigg)$$

Thus, if

$$B_j = \left[1 + \frac{n}{r(K-j)-1}\right]\left[1 + \frac{n}{r(K-j)-2}\right]\cdots\left[1 + \frac{n}{r(K-j)-r}\right]\lambda p^r$$

and

$C_0 = 1$, then

$C_1 = 1 + B_0 C_0$

$C_{j+1} = 1 + B_j C_j$

and

$$A_K = \frac{(r+n-1)!}{(r-1)!n!}\left(1 + B_{K-2}\, C_{K-2}\right)$$

Now that the distribution of demand during lead time and methods for calculating the probabilities have been discussed, it is a simple matter to find the number of items of stock that are required as the reorder point. The reorder point must be the number of units of stock that will last until new replacement stocks arrive with the probability of a stockout during the lead time small.

Let a = Acceptable probability of a stockout during lead time

Then

$$P\left[Z > R\right] \leqslant a$$

must be solved for the value of R.

$$1-P\left[Z>R\right] = P\left[Z=0\right] + P\left[Z=1\right] + \dots + P\left[Z=R\right]$$

$\therefore$ R is the smallest number such that

$$\sum_{n=0}^{R} P\left[Z = n\right] \geqslant 1 - a$$

5. Negative Binomial Distribution of Demand During Lead Time

It would be very helpful if the distribution function of demand during lead time could be found for any distribution functions of demand and lead time in a familiar form, but this is not possible. It can be done in some special cases. One known case is given below as an example. Consider deriving the mass function of demand during lead time where:

(1) The demand is random and the number of demands d during a fixed interval of time t has a Poisson distribution.

$$p(d;at) = \frac{e^{-at} \quad (at)^d}{d!} \qquad ,d = 0,1,2,\ldots$$

where a is the average number of demands per unit time.

(2) The lead time, t, has a gamma type probability density function

$$f(t;u,k) = \frac{ue^{-ut} \quad (ut)^{k-1}}{(k-1)!} \qquad , t \geqslant 0$$

with mean $\frac{k}{u}$ and variance $\frac{k}{u^2}$

Then the probability of exactly z demands during the lead time is

$$p(z;k,u,a) = \int_0^\infty \frac{e^{-at} \quad (at)^z}{z!} \cdot \frac{ue^{-ut} \quad (ut)^{k-1}}{(k-1)!} \, dt$$

$$= \frac{a^z u^k}{z!(k-1)!} \int_0^\infty t^{z+k-1} e^{-(a+u)t} dt$$

Letting $x = (a+u) t$, then

$$p(z;k,u,a) = \frac{a^z u^k}{(a+u)^{z+k} z!(k-1)!} \int_0^\infty x^{z+k-1} e^{-x} dx$$

$$= \frac{a^z u^k}{(a+u)^{z+k}} \cdot \frac{(k+z-1)!}{z!(k-1)!}$$

$$= \binom{k+z-1}{z} \left(\frac{u}{a+u}\right)^k \left(\frac{a}{a+u}\right)^z$$

the last step being valid if k is an integer. Thus the demand during lead time has a negative binomial distribution with

$$\text{mean} = \frac{ak}{u}$$

and

$$\text{variance} = \frac{ak}{u}\left(1 + \frac{a}{u}\right)$$

6. Summary and Conclusions

The distribution of demand during lead time has been discussed when discrete random variables describe the demand and lead time. It is unfortunate that the distributions do not appear as well known distributions. The use of probability generating functions has been shown as a method by which the distribution can be found and used. Calculations of the probabilities are cumbersome since in the examples shown each of coefficients of the s^k in the probability generating function involved an infinite sum. The amount of computer time required for the calculations has not been investigated, but since the calculations are straight forward and the number of terms required before the infinite sum converges to an approximate value are relatively small, the use of the procedure outlined in Section 4 should be useful and not too cumbersome to preclude their use.

The use of the generating function method requires data for the estimation of the parameters in the demand and lead time distributions. This data is readily available in the historical files of an inventory system. Other methods of estimating the demand during lead time requires collection of data directly from the number of demands during a lead time. This data is not usually collected and even if it were available sufficient data to obtain a good statistical sample would require a very long time.

Further investigation of distributions calculated by the methods in Section 4 may show that the probability distributions could be fit to some known distributions and the resulting distribution used to approximate the demand during lead time.

This investigation has been made primarily to consider a method of determining the demand during lead time distributions for low demand items. Low demand items are particularly troublesome in an inventory system. Military spare parts inventories have large numbers of very low demand items. The low demand items are also characterized by high variances [6].

The use of the Poisson and negative binomial distributions have been used extensively in this paper since these distributions have been found to fit low and medium demand items. The Poisson distribution is generally used for items with a demand of two or less items per year. The negative binomial distribution has been used for items with demands ranging from two to one hundred per year [7]. In the case where the Poisson demand distribution is applicable, the generating function method for establishing a reorder point can be readily applied. Since the reorder point will be relatively low, the number of probabilities that must be calculated before the sum is equal to or greater than the service level desired will be small. It has not been discussed in this paper, but when reorder points have been previously established, the probability generating function for demand during lead time can be used to find the tail of the distribution. With a reorder point previously established, a good estimate of the value of reorder point is available and with several trials on either side of the old reorder point, the tail of the distribution could be used to determine the new order point. In this manner, the probabilities would not have to be summed until the sum was greater than the desired probability. Feller [2]

gives the following relations between the probability generating function and the generating function of the tail of the distribution.

Let $Q(s)$ = the generating function of the tail

Then

$$Q(s) = \frac{1 - h(s)}{1 - s}$$

The calculation of the tail in this paper has not been investigated since it gets more complex than the probability generating function. However, in special cases and in a particular situation as described above, the tail of the distribution may be the easiest method for setting a reorder point.

The special case of Poisson type demand and gamma distributed lead times was discussed because this combination of demand and lead time distributions lead to a negative binomial distribution of demand during lead time. Further investigation of this model with actual data may be very fruitful. The Poisson distribution is a good distribution for low demand items and the gamma distribution is a very versatile two parameter distribution that can be used to approximate a wide range of lead time patterns [5]. Consequently, any demand and lead time process that may possibly be fit to this model should be considered, especially since the probability law of demand during lead time may be readily computed.

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APPENDIX I

MEAN AND VARIANCE FROM PROBABILITY GENERATING FUNCTIONS

The mean and variance of a discrete random variable may be calculated from the probability generating function. It may be shown that (see e.g., Feller [2])

$$E(X) = f'(1)$$

$$\mathrm{Var}(X) = f''(1) + f'(1) - [f'(1)]^2$$

As an example, the mean and variance of demand during lead time when lead time is a translated Poisson random variable

$$p(x;\lambda) = e^{-\lambda} \frac{\lambda^{x-1}}{(x-1)!} \qquad , \ x = 1,2,\cdots$$

with mean = $(\lambda + 1)$ and variance = λ

and demand has the negative binomial distribution

$$p(x;r,p) = \binom{r+x-1}{x} p^r q^x \qquad , \ x = 0,1,2,\cdots$$

with mean = $\frac{r\,q}{p}$ and variance = $\frac{r\,q}{p^2}$

then the random variable Z has a probability generating function

$$h(s) = \left(\frac{p}{1-qs}\right)^r \exp\left\{\lambda\left[\left(\frac{p}{1-qs}\right)^r - 1\right]\right\}$$

taking the first derivative with respect to s we have

$$h'(s) = rqp^r\left(\frac{p}{1-qs}\right)^{r+1} \exp\left\{\lambda\left[\left(\frac{p}{1-qs}\right)^r - 1\right]\right\} + rqp^{2r}\lambda\left(\frac{1}{1-qs}\right)^{2r+1} \exp\left\{\lambda\left[\left(\frac{p}{1-qs}\right)^r - 1\right]\right\}$$

and

$$h'(1) = \frac{rq}{p} + \frac{rq\lambda}{p}$$

$$= \frac{rq}{p}(1+\lambda)$$

the second derivative evaluated at one is

$$h''(1) = \frac{1}{p^2}\left\{\left[rq(1+\lambda)\right]^2 + r^2q^2\lambda + rq^2(1+\lambda)\right\}$$

then

$$\mathrm{Var}(Z) = h''(1) + h'(1) - \left[h'(1)\right]^2$$

$$= \left(\frac{rq}{p}\right)^2(1+\lambda)^2 + \left(\frac{rq}{p}\right)^2\lambda + \frac{rq^2}{p^2}(1+\lambda) + \frac{rq}{p}(1+\lambda) - \left(\frac{rq}{p}\right)^2(1+\lambda)^2$$

$$= \frac{rq}{p^2}(1+\lambda) + \left(\frac{rq}{p}\right)^2$$

These results can be verified against the general formulas for the mean and variance given in Section 2.

[redacted]

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